II. Portfolio Selection

1. Introduction

- Our goal in this topic is to find the best portfolio that if feasible for an investor
- We will follow a top–down analysis of portfolio construction:
  - Capital allocation decision: the choice between risk-free and risky assets
  - Asset allocation decision: the distribution of risky investments across different classes of assets (e.g., stock, bonds, real estate etc.)
  - Security selection decision: the choice of particular securities to be held within each asset class.
- We start from capital allocation problem between risk–free asset and the risky portion of the portfolio in which the composition of the risky portion is taken as given.
  - To solve the problem first we determine the risk–return tradeoff
  - Then show how risk aversion determines the optimal mix of risky and risk–free assets
- We do not distinguish between different classes of risky assets. That is, we skip asset allocation
- In security selection we consider two risky securities and then extend the analysis to more than two risky securities
- We consider only static model of trading:
  - We assume that an investor trades only once (today) and receives payoffs in the future

- Our analysis is called mean–variance analysis since it is based on mean–variance preferences

- Finally, we consider an investor in a competitive market so he/she is a price–taker and cannot move prices

2. Capital Allocation across Risky and Risk–Free assets

- We consider a complete portfolio (called C) that is comprised from a risk–free asset and a risky portfolio (called P)

- Let us assume that risky portfolio is given. Our goal is to find the fraction of the complete portfolio to be invested in the risky portfolio versus the risk–free asset.

- **The main assumption**: when we shift wealth from the risky portfolio to the risk–free asset, we do not change the relative proportions of various assets within the risky portfolio.

  – The assumption is made for convenience and requires that risk and return characteristics of securities do not change as we shift wealth

- **Example**. Suppose you are an investment advisor and a client of yours has a portfolio worth $\$1,000,000$. Of this amount $\$250,000$ are invested in the risk-free money market fund, $\$300,000$ are invested in the Stock Fund and the remainder in the Bond Fund.

- Holdings of Stock Fund and Bond Fund comprise her risky portfolio

- Question: what are the weights of each of these funds in the risky portfolio?

  (a) the total value of her risky portfolio =

  (b) $w_S =$

  (c) $w_B =$

- Since your client is worried about recent market performance, she calls you up and tells you she wants to increase her investment in the money market fund to 60% of her complete portfolio.

- Question: how much of her holdings in Stock and Bond Funds will your client have to sell to achieve 60% weight in the risk-free money market fund?

  (a) her new investment in the money market fund will be $0.6 \times \$1,000,000 = \$600,000$ and the remainder, $\$400,000$ will be in the risky portfolio.

  (b) so she will have to sell worth of her holdings in the risky funds.

  (c) since the relative proportions of various assets within the risky portfolio remain unchanged, she will have to sell of her holdings in the Stock Fund and of her holdings in the Bond Fund.

- Since expected return and standard deviation of the return of a risky portfolio are fixed, we will also call this portfolio as a risky asset or risky security

Investment Opportunity Set

- Now let us examine the risk–return combinations available to investors

- Consider a complete portfolio C that includes the following two assets:

  - T-bill with return r_f
  - risky asset P with expected return and standard deviation of return $E(r_P)$ and σ_P, respectively

- Assume that

  - y is a fraction of portfolio C in asset P and
  - $1 - y$ be a fraction of portfolio C in T–bills

- Expected return and standard deviation of portfolio C are

$$E(r_C) = yE(r_P) + (1-y)r_f = r_f + y[E(r_P) - r_f]$$

$$\sigma_C = y\sigma_P$$

It follows from the last formula that $y = \sigma_C/\sigma_P$. Hence

$$E(r_C) = r_f + \frac{E(r_P) - r_f}{\sigma_P}\,\sigma_C \tag{2-1}$$

- The last equation presents all possible combinations of $E(r_C)$ and σ_C that are available to an investor. The set of these combinations is called the *investment opportunity set.*

- Because (2-1) is an equation of a straight line on the $\sigma_C - E(r_C)$ plane then so is the investment opportunity set.

- The straight line is called the **capital allocation line (CAL)**.

- The slope of the CAL is given by

$$S = \frac{E(r_P) - r_f}{\sigma_P}.$$

It equals the increase in the expected return of the chosen portfolio per unit of additional standard deviation.

- S is called the **reward–to–variability ratio** (also **Sharpe ratio**).

- **Example**: Suppose we have the following characteristics of T–bill and asset P

	T-bill	asset P
$E(r)$	0.05	0.15
σ	0	0.25

It follows that the reward–to–variability ratio is $S = \frac{E(r_P)-r_f}{\sigma_P} = \frac{0.15-0.05}{0.25} = 0.4$.

Let us find $E(r_C)$ for a few different values of σ_C:

- $\sigma_C = 0$, then

  $y = \sigma_C / \sigma_P = 0$

  $E(r_C) = r_f + \frac{E(r_P) - r_f}{\sigma_P} \sigma_C = r_f = 0.05$

- $\sigma_C = 0.125$, then

  $y = 0.5, \quad E(r_C) = 0.05 + 0.4 \times 0.125 = 0.1$

- $\sigma_C = 0.25$, then

  $y = 1.0, \quad E(r_C) = 0.05 + 0.4 \times 0.25 = 0.15$

- $\sigma_C = 0.375$, then

  $y = 1.5, \quad E(r_C) = 0.05 + 0.4 \times 0.375 = 0.20$

- The found risk–return combinations are shown on the following graph.

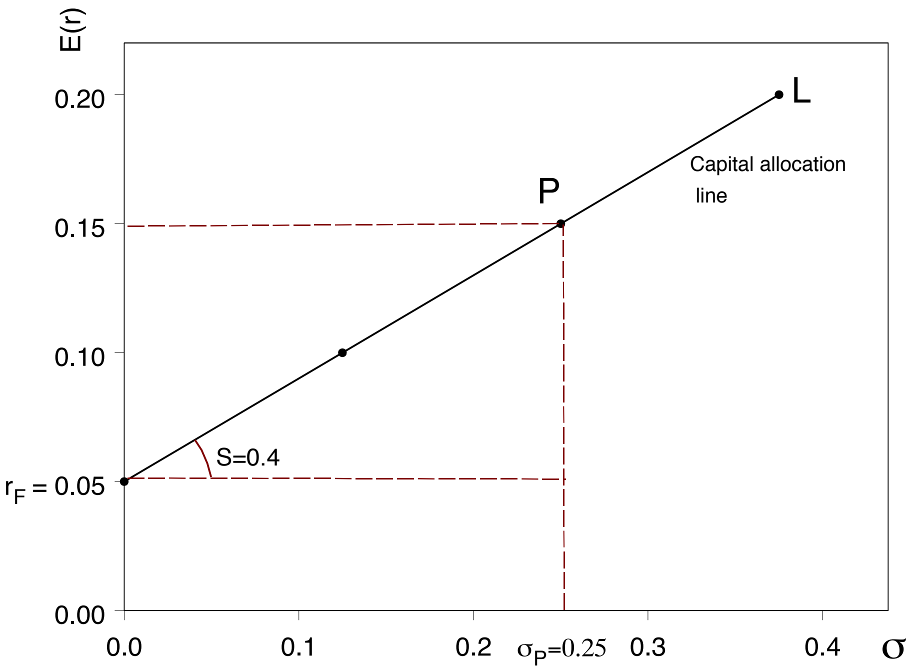


Figure 1: Capital Allocation Line

- Note that an investor allocates all her wealth to risky assets at point P of the CAL ($y = 1$).
- At the allocations lying to the right from point P an investor takes a leveraged position in the risky portfolio. That is, she borrows in the money market to buy more risky assets.
  - Consider the last example for $y = 1.5$ (point L on the graph). An investor borrows cash (sells T–bills) equal half of her portfolio value ($1 - y = -0.5$) and forward the proceeds to buy risky assets
  - This strategy yields higher return but is also more risky. Moreover, the return–to–variability ratio remains the same

Practice Problem

Suppose you manage a risky portfolio with an expected rate of return of 20% and a standard deviation of 40%. The T-bill rate is 5%. Also, suppose your risky portfolio includes the following assets in the given proportions:

Stock X	20%
Stock Y	50%
Stock Z	30%

Suppose your client comes to you and asks you to help her create a portfolio with expected return of 15%.

- What fraction (y) will she need to invest in your risky portfolio to obtain this expected return?

- What is the standard deviation of the rate of return on your client's portfolio?

- What are the investment proportions of your clients complete portfolio, including the position in T -bills?

  $y_X =$

  $y_Y =$

  $y_Z =$

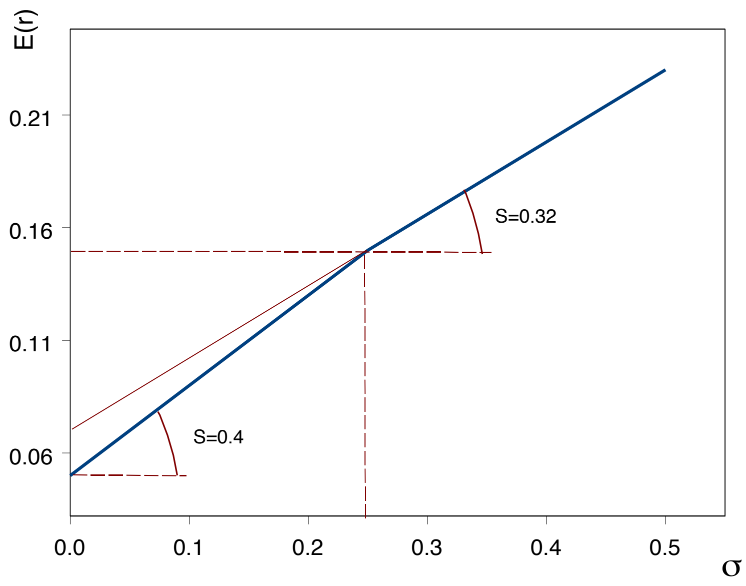
  $y_{T-bill} =$

- In practice, investors have borrowing rate that is higher than a rate for lending money. Therefore, the investment opportunity set is different from that in Figure 1
  - Use the example on page 20 and assume that the borrowing rate is 7%
  - Because investor borrows at allocations lying to the right from point P on CAL and lends at allocations lying to the left from P, the slope of CAL decreases after P and becomes (see also the graph below)

$$S = \frac{E(r_P) - r_f^B}{\sigma_P},$$

where r_f^B is a borrowing rate. In the case of our example

$$S = \frac{0.15 - 0.07}{0.25} = 0.32$$



- We now have established the feasible set of all complete portfolios available to an investor. The next section will answer the question: Which portfolio an investor will pick?

Capital Allocation Problem

- Recall that utility that investor derives from a given complete portfolio can be described by the expected return and variance of this portfolio. In particular, we used the following utility function:

$$U = E(r_C) - \frac{1}{2}A\sigma_C^2$$

- The goal of an investor is to find the optimal complete portfolio C. To achieve this goal, she maximizes her utility function by finding optimal proportion $y^\star$ of risky portfolio P in complete portfolio C

  – Recall that the expected return of complete portfolio is

$$E(r_C) = r_f + y[E(r_P) - r_f]$$

and its standard deviation is

$$\sigma_C^2 = y^2\sigma_P^2$$

– We substitute the last constraints into the utility function from the complete portfolio and find the capital allocation problem faced by an investor

$$\max_y \{r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2\}$$

– We find from the first order condition

$$[E(r_P) - r_f] - Ay\sigma_P^2 = 0 \ \rightarrow \ Ay\sigma_P^2 = E(r_P) - r_f$$

or,

$$y^\star = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

- The solution shows that the optimal position in the risky asset is inversely proportional to the level of risk aversion and the level of risk and directly proportional to the risk premium offered by the risky asset.
- The maximal expected utility is achieved at $y = y^\star$:

$$\begin{aligned} U_{max} &= r_f + y^\star[E(r_P) - r_f] - \frac{1}{2}Ay^{\star 2}\sigma_P^2 = r_f + \frac{[E(r_P) - r_f]^2}{A\sigma_P^2} - \frac{1}{2}\frac{[E(r_P) - r_f]^2}{A\sigma_P^2} \\ &= r_f + \frac{[E(r_P) - r_f]^2}{2A\sigma_P^2} \end{aligned}$$

- It follows that the higher the reward–to–variability ratio of a risky asset the higher the utility of an investor

Example:

Suppose an investor has a coefficient of risk aversion, A, equal to 2. Then if she has an access to the following assets

her optimal position in the risky asset would be:

$$y^\star = \frac{0.15 - 0.05}{2 \times 0.25^2} = 0.8 = 80\%$$

	asset F	asset P
$E(r)$	0.05	0.15
σ	0	0.25

The expected return and standard deviation of her complete portfolio then would be:

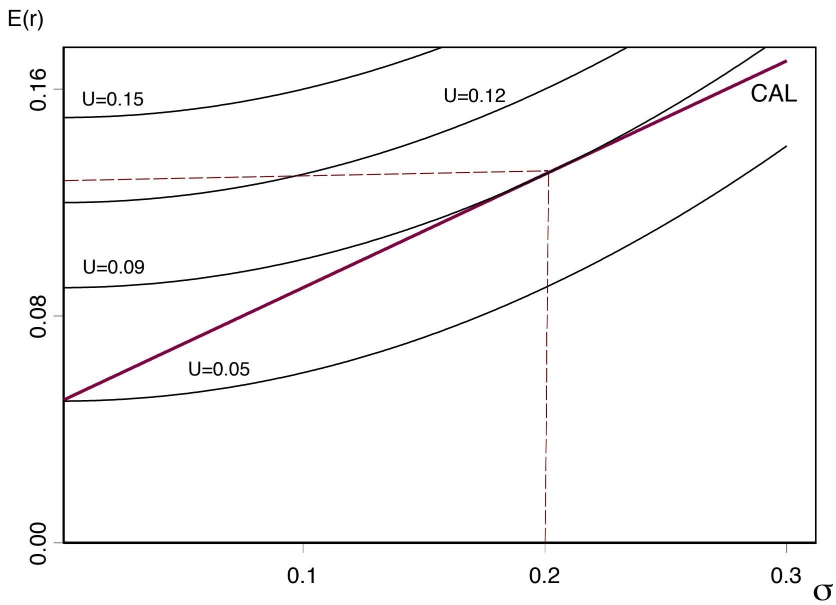
$$E(r) = 0.05 + 0.8(0.15 - 0.05) = 0.13$$

$$\sigma_r = \sqrt{0.8^2 0.25^2} = 0.2$$

Allocation $y^\star$ provides the utility

$$U = 0.13 - \frac{1}{2} 2 \times 0.2^2 = 0.09$$

- Capital allocation problem can also be solved graphically:
  - Draw the investment opportunity set (CAL)
  - Draw an indifference curve that is tangent to CAL. Point of tangency gives us optimal σ_r and the solution of the problem since $y = \sigma_r / \sigma_P$
- Question: The graph above shows the optimal portfolio for an investor with A=2. Find an optimal portfolio graphically for an investor who is more risk averse (say, with A=4).
- Occasionally, risk premium can be negative.
  - Then, an investor will take a short position in the stock ($y^\star < 0$)
  - Shorting stock means borrowing its shares from another investor and then selling them. An investor has to return the shares later
  - An investor shorts a stock if she thinks that its price is going to fall



– If a lot of investors short a stock then its price will fall and the expected return increase

- **Assumption: We assume in this course that stock positions of an investor are unrestricted by market regulations unless specified otherwise**
- In particular, an investor can take arbitrary short position in a stock
  – In practice, an investor cannot take arbitrary short position since she has to maintain margin account. We neglect this regulation for simplicity
- Question: Is short–selling risky? Why?

Practice Problem

Suppose a stock fund has an expected return of 12% and standard deviation of 20%. The T-bill rate is 3%, while the borrowing rate faced by investors is 6%.

There are two investors Mary and Michael with coefficients of risk aversion 5 and 1, respectively.

- What are optimal proportions in risky stock fund by each investor?

  Mary:

  $y^\star =$

  Michael:

  $y^\star =$

- What are the expected returns and standard deviations of their portfolios?

  Mary:

  $E(r) =$

  $\sigma_r =$

  Michael:

  $E(r) =$

  $\sigma_r =$

- What are the utilities provided by these portfolios?

  Mary:

  $U =$

  Michael:

  $U =$

Passive strategies and capital market line

- Determination of assets in portfolio P may result from a passive or an active strategy

- A **passive strategy** describes a portfolio decision that avoids any direct or indirect security analysis
- A natural candidate for a passively held risky asset would be a well–diversified portfolio of common stocks. For example, a market index, such as TSE 300
- Definition: the capital allocation line provided by T–bills and a broad index of common stocks is called the **capital market line** (CML).
- A passive strategy might seem naive but there are arguments in its favor:
  - Active strategy is not free
    - time investment if you are doing your own stock research
    - fees charged by an investment professional. For example, the average annual fee by mutual funds for managing portfolio with Canadian equities is about 2.5% of total investment
  - free rider benefit
    - if there are some active, knowledgeable investors who do research and trade on the findings of their research, these findings will be reflected in the stock price;
    - in other words, these investors will quickly bid up prices of undervalued stocks and bid down (by selling) prices of overvalued stocks, so that most stocks in the market are fairly priced;
    - this implies that a market portfolio is expected to be a fair buy;
    - so *free rider* benefit refers to the fact that you can ride on someone else's research for free.

Diversification and portfolio risk

- We find two broad sources of uncertainty in an individual stock

- First, there is a firm–specific risk
  - Example: risk that a manager of company will retire, risk that there will be a merger with another company
  - This risk can be completely removed from portfolio by means of diversification. It is called **firm–specific risk**, or **nonsystematic risk**, or **diversifiable risk**
  - A firm specific risk for a given stock is independent from a firm specific risk of any other stock
- Second, there is the risk resulting from the conditions in the general economy
  - Example: risk from business cycles, inflation rate, interest rate, and exchange rate
  - This risk is present in almost all risky securities and cannot be diversified away. It is called **market risk**, or **systematic risk**, or **nondiversifiable risk**
  - Systematic risk is independent from a nonsystematic risk of any security

Practice Problem

Suppose you manage an active mutual fund with an expected rate of return of 16% and a standard deviation of 25%. The T-bill rate is 4%. Your only competitor is an index fund that has an expected return of 12% and a standard deviation of 20%. What are the slopes of CAL of an active mutual fund and CML?

$S_{CAL} =$

$S_{CML} =$

What is the maximum fee you could charge (as a percentage of the investment deducted at the end of the year) that would still leave your client indifferent between the index fund and your fund? (hint: when will your client be indifferent between your fund and the index fund?)

3. Portfolio of Two Risky Assets

- Now let us consider security selection for the case when an investor has an access to only two risky assets (no risk–free asset!)
- Suppose you can invest in two risky assets which returns have the following statistics

	asset 1, %	asset 2, %
$E(r)$	8	13
σ_r	12	20
$Cov(r_1, r_2)$	0.0072	
ρ	0.30	

- The expected return of your portfolio is

$$E(r_p) = w_1 E(r_1) + w_2 E(r_2),$$

where r_1, r_2 and w_1, w_2 are rates of return of and proportions of funds invested in asset 1 and 2, respectively

  - It follows that the expected return of the portfolio is a weighted average of expected return of component securities

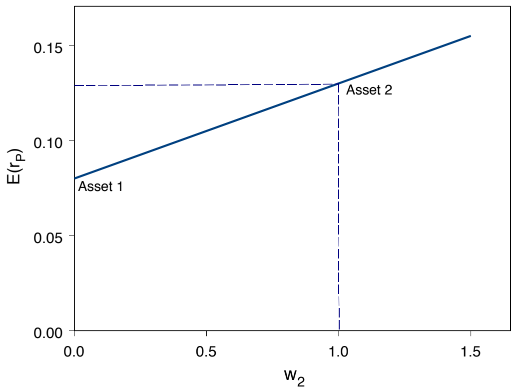
– Because $w_1 = 1 - w_2$, we find

$$E(r_p) = E(r_1)+w_2[E(r_2)-E(r_1)] = 0.08+w_2(0.13-0.08) = 0.08+0.05\,w_2$$

– The latter is an equation of the straight line passing through two points:

$$w_2 = 0, \quad E(r_p) = 0.08 + 0.05 \times 0 = 0.08$$

$$w_2 = 1, \quad E(r_p) = 0.08 + 0.05 \times 1 = 0.13$$



- Question: Can an investor make the expected return of her portfolio to be more than 13%? If yes, then how?

- Risk. Let us assume that an investor does not short any of the two risky securities

  – Portfolio variance

$$\sigma_P^2 = w_1^2 \times \sigma_1^2 + w_2^2 \times \sigma_2^2 + 2w_1w_2\,Cov(r_1, r_2),$$

or

$$\sigma_P^2 = (1 - w_2)^2 \times \sigma_1^2 + w_2^2 \times \sigma_2^2 + 2(1 - w_2)w_2\,Cov(r_1, r_2),$$

where σ_1 and σ_2 are standard deviations of returns of asset 1 and 2, respectively, and

$$Cov(r_1, r_2) = \sigma_1\sigma_2\rho$$

– If $\rho = 1$, then

$$\sigma_P = \sqrt{w_1^2 \times \sigma_1^2 + w_2^2 \times \sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2} = \sqrt{[w_1\sigma_1 + w_2\sigma_2]^2} = w_1\sigma_1 + w_2\sigma_2$$

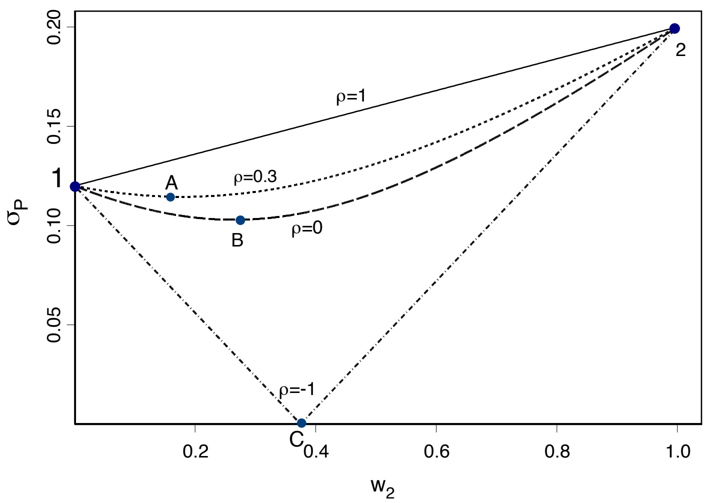
That is, the standard deviation of the portfolio with perfect positive correlation is just weighted average of the component standard deviations.

– It follows that the expected return of a portfolio increases linearly with w_2 while the standard deviation increases less than linearly with w_2

– We conclude that portfolios of less than perfectly correlated assets always offer better risk–return opportunities than individual component securities on their own

– Therefore, an investor is better of diversifying

– An investor can change the risk of her portfolio at fixed σ_1, σ_2 and ρ by varying the wight w_2. The following figure shows σ_P versus w_2 for different values of ρ.



– The smaller the correlation, the greater the risk reduction potential

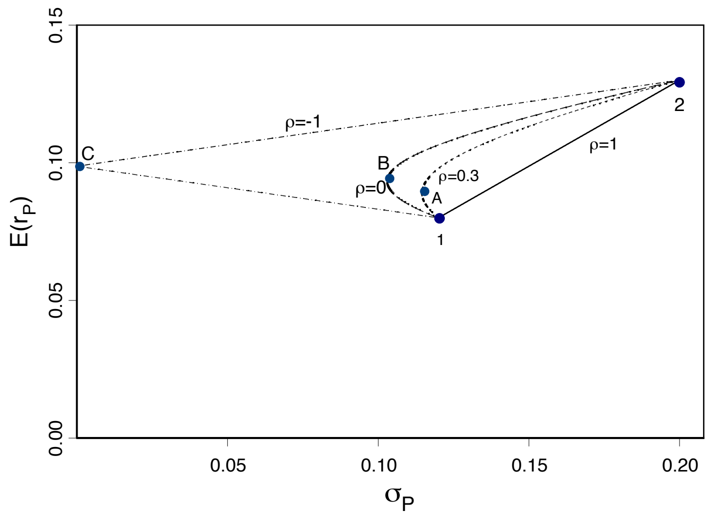
– If $\rho = -1.0$, we can create a portfolio with zero variance (standard deviation);

- If $\rho = 1.0$ and short–selling is not allowed, no risk reduction is possible (that is, the risk of stock one is smaller than the risk of portfolio with the two stocks).
- If short–selling is allowed then the lines on the figure above should be extended beyond points 1 and 2.
- It follows that that our main conclusions on the role of diversification hold even if short–selling is allowed
- Question: Assume that $\rho = 1$ and short–selling is allowed, can an investor make her portfolio risk free? If yes, then how?

- The **minimum–variance portfolio** is a portfolio with minimal standard deviation (see portfolios A, B and C in the figure above)
- The minimum–variance portfolio has a standard deviation smaller than that of either of individual component assets

- Let us combine the last two figures to show the relationship between the portfolio's level of risk and the expected return
- Each line in the figure below shows the **portfolio opportunity set** for the corresponding correlation coefficient. It gives the risk–return combinations that can be constructed from the two available assets.
- Question. Investments opportunities sets shown on the figure below assume no short–selling. How do they change if short–selling is allowed?

- Question: Assume that you can choose between different markets with two risky



securities that differ only by their correlation coefficients. Which correlation coefficient you would like to have?

- To find an optimal portfolio an investor has to maximize her expected utility by choosing a feasible risk–return combination from a portfolio opportunity set

Capital Allocation Across Two Risky Securities

- Suppose an investor can invest in two risky assets that have the following the rate–to–return distribution

	asset 1, %	asset 2, %
$E(r)$	8	13
σ_r	12	20
$Cov(r_1, r_2)$	0.0072	
ρ	0.30	

The coefficient of the risk–aversion of an investor, A, is 4. Let us find an optimal portfolio for this investor.

An investor maximizes her utility function

$$U = E(r_P) - \frac{1}{2}A\sigma_P^2$$

by finding optimal risk exposure (proportion $w_2^\star$) from the portfolio opportunity set

- Recall that

  $$E(r_p) = E(r_1) + w_2[E(r_2) - E(r_1)]$$

  and

  $$\sigma_P^2 = (1 - w_2)^2 \times \sigma_1^2 + w_2^2 \times \sigma_2^2 + 2(1 - w_2)w_2\, Cov(r_1, r_2)$$

- We substitute the last constraints into the utility function to find the capital allocation problem faced by an investor

  $$\max_{w_2}\left\{E(r_1)+w_2[E(r_2)-E(r_1)]-\frac{1}{2}A[(1-w_2)^2\times\sigma_1^2+w_2^2\times\sigma_2^2+2(1-w_2)w_2\, Cov(r_1,r_2)]\right\}$$

- Taking derivative of the utility function and setting it to zero results in

  $$[E(r_2)-E(r_1)]-A[-(1-w_2)\times\sigma_1^2+w_2\times\sigma_2^2+(1-2w_2)\, Cov(r_1,r_2)] = 0 \;\; \rightarrow$$

  $$Aw_2[\sigma_1^2 + \sigma_2^2 - 2Cov(r_1, r_2)] = [E(r_2) - E(r_1)] + A[\sigma_1^2 - Cov(r_1, r_2)]$$

  or,

  $$w_2^\star = \frac{E(r_2) - E(r_1) + A[\sigma_1^2 - Cov(r_1, r_2)]}{A[\sigma_1^2 + \sigma_2^2 - 2Cov(r_1, r_2)]}. \tag{2-2}$$

- Thus, in our example,

  $$w_2^\star = \frac{0.13 - 0.08 + 4(0.12^2 - 0.0072)}{4[0.12^2 + 0.2^2 - 2 \times 0.0072]} = 0.4925.$$

  $$w_1^\star = 1 - w_2^\star = 0.5075$$

– The expected return, the standard deviation and utility of the resulting portfolio are

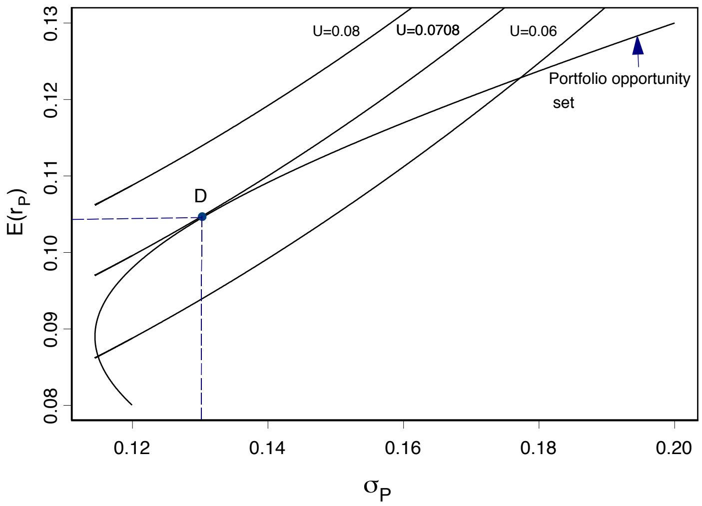
$$E(r_p) = E(r_1) + w_2^\star[E(r_2) - E(r_1)] = 0.08 + 0.4925(0.13 - 0.08) = 0.1046$$

$$\sigma_P = \sqrt{w_1^{\star 2} \times \sigma_1^2 + w_2^{\star 2} \times \sigma_2^2 + 2w_1^\star w_2^\star Cov(r_1, r_2)} =$$

$$\sqrt{0.5075^2 \times 0.12^2 + 0.4925^2 \times 0.2^2 + 2 \times 0.5075 \times 0.4925 \times 0.0072} = \sqrt{0.0170} = 0.13$$

$$U = E(r_P) - \frac{1}{2}A\sigma_P^2 = 0.1046 - 0.5 \times 4 \times 0.13^2 = 0.0708$$

– Graphical solution of the problem is shown in the following figure



- Assume that correlation coefficient between two risky securities is -1. How you can solve capital allocation problem in this market without applying formula (2-2)?

Practise problem

(a) What are the optimal weights w_2 in the above example if $\rho = -0.5$?

$w_2^\star =$

$w_1^\star =$

(b) What are the expected return, the standard deviation and utility of the resulting portfolio?

$E(r_p) =$

$\sigma_P =$

$U =$

(c) How your results are different from those in the case of $\rho = 0.3$? Why?

4. Allocation between Two Risky and One Riskless Assets

- Suppose that an investor with a known coefficient of risk aversion can trade the two risky assets
- Now suppose that in addition to the two risky assets an investor can invest in risk–free T–bills yielding 5%

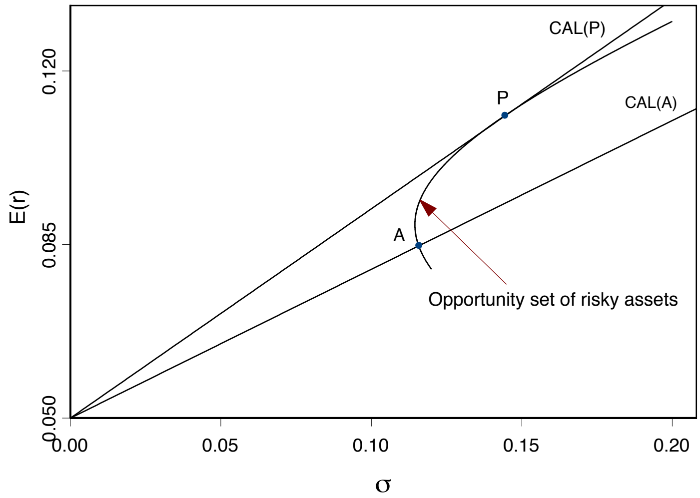
	asset 1, %	asset 2, %
$E(r)$	8	13
σ_r	12	20
$Cov(r_1, r_2)$	0.0072	
ρ	0.30	

- Our goal is to find a complete portfolio C which is comprised from T-bills and assets 1 and 2
- Like in the case with one risky and one riskless assets, the investment opportunity set is going to be a straight line. However, this time we also have to find the portfolio that includes only risky assets
- Therefore, finding the optimal portfolio will involve two steps:
  - Finding the optimal portfolio P that includes only risky assets
  - Finding the optimal fraction of the complete portfolio to invest in the portfolio P

Step 1. Finding the optimal risky portfolio P

- First, following the steps outlined in the previous section, we construct the portfolio opportunity set assuming that only two risky assets can be traded
- We then draw a CAL line from the risk–free rate (5% in our case) and look for the CAL that has the highest possible slope. (Recall that an investor achieves the highest utility for the highest reward–to–variability ratio)
- CAL with the highest possible slope is tangent to portfolio opportunity set obtained from the two risky assets. (Compare CAL's through points A and P in the figure below)

- Portfolio P is given by the point of tangency of the CAL with the highest possible slope and the portfolio opportunity set



- Let us find portfolio P mathematically

  – We want to find weights $w_1^\star$ and $w_2^\star$ that maximize slope of the CAL:

$$\max_{w_1, w_2} S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

where

$$E(r_P) = w_1 E(r_1) + w_2 E(r_2),$$

$$\sigma_P = \sqrt{w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 Cov(r_1, r_2)}$$

$$w_1 + w_2 = 1.$$

Or,

$$\max_{w_2} \frac{(1 - w_2) E(r_1) + w_2 E(r_2) - r_f}{\sqrt{(1 - w_2)^2 \sigma_1^2 + w_2^2 \sigma_1^2 + 2(1 - w_2) w_2 \, Cov(r_1, r_2)}}$$

Taking derivative with respect to w_2 and setting the derivative equal to zero, we find

$$\begin{aligned} w_2^\star &= \frac{S_{P2} - \rho_{12} S_{P1}}{S_{P2} + \frac{\sigma_2}{\sigma_1} S_{P1} - \rho_{12}(S_{P1} + \frac{\sigma_2}{\sigma_1} S_{P2})} \\ w_1^\star &= 1 - w_2^\star \end{aligned}$$

where S_{P1}, S_{P2} are the Sharpe ratios of the first and second assets, respectively.

– Substituting our data into the last equation gives

$$
\begin{aligned}
S_{P1} &= (0.08 - 0.05)/0.12 = 0.25, \quad S_{P2} = (0.13 - 0.05)/0.20 = 0.4 \\
w_2^\star &= \frac{0.4 - 0.25 \times 0.3}{0.4 + 0.25 \times 0.20/0.12 - 0.3(0.25 + 0.4 \times 0.2/0.12)} = 0.6 \\
w_1^\star &= 1 - 0.6 = 0.4 \\
E(r_P) &= 0.4 \times 0.08 + 0.6 \times 0.13 = 0.11 \\
\sigma_P &= \sqrt{0.4^2 0.12^2 + 0.6^2 0.2^2 + 2 \times 0.4 \times 0.6 \times 0.0072} = 0.142 \\
S_P &= \frac{0.11 - 0.05}{0.142} = 0.42
\end{aligned}
$$

- Question: How portfolio P depends on risk aversion of an investor?

- It follows that if investors have homogeneous beliefs, that is their expectations on future returns are the same, then each investor buys the same risky portfolio P.

- The latter statement comprises the **two fund separation theorem** which states that if investors have the same beliefs then they buy the same risky portfolio no matter what is a wealth and risk aversion of each investor are

- If holds, then the two fund separation theorem would substantially simplify the life for mutual funds

Step 2. Finding capital allocation to portfolio P

- An investor is going to allocate her capital between risk–free T–bills and risky portfolio P.

- We know from our previous analysis that the optimal proportion of wealth, $y^\star$, to be allocated to portfolio P is

$$y^\star = \frac{E(r_P) - r_f}{A\sigma_P^2},$$

where A is a coefficient of risk aversion.

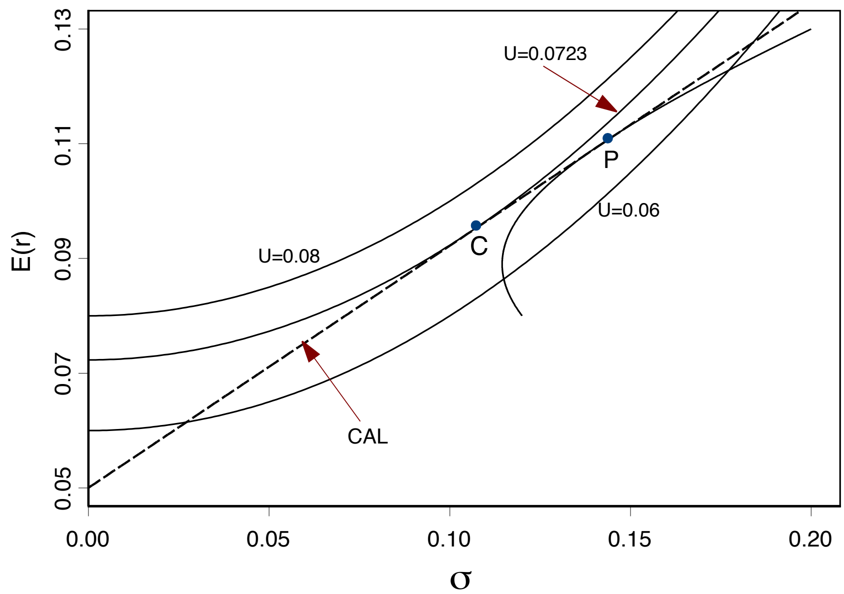
- Assuming that $A = 4$, we find

$$y^\star = \frac{0.11 - 0.05}{4 \times 0.142^2} = 0.7439$$

  - That is, an investor will invest 74.39% of her wealth in portfolio P and 25.61% in T–bills.

  - Moreover, she will invest $74.39 \times 0.4 = 29.76\%$ of her wealth in risky asset 1 and $74.39 \times 0.6 = 44.63\%$ in risky asset 2

- Graphical solution of the asset allocation problem is shown below



Practice Problem

There are two securities: stock G and stock H. Stock G has expected return of 12% and standard deviation of 20%, while stock H has expected return of

3.5% and standard deviation of 4%. The correlation coefficient between the two securities is 0.35. The riskfree rate is 3%.

1. What is the weight of stock G in the optimal risky portfolio?

  – What does a negative weight in stock H mean?

  – In general, an investor can take short position in a stock even if its risk premium is positive. This allows her to improve diversification. That is, an investor shorts a stock to forward the proceeds to buy more attractive stock and increase the reward–to–variability ratio of her risky portfolio

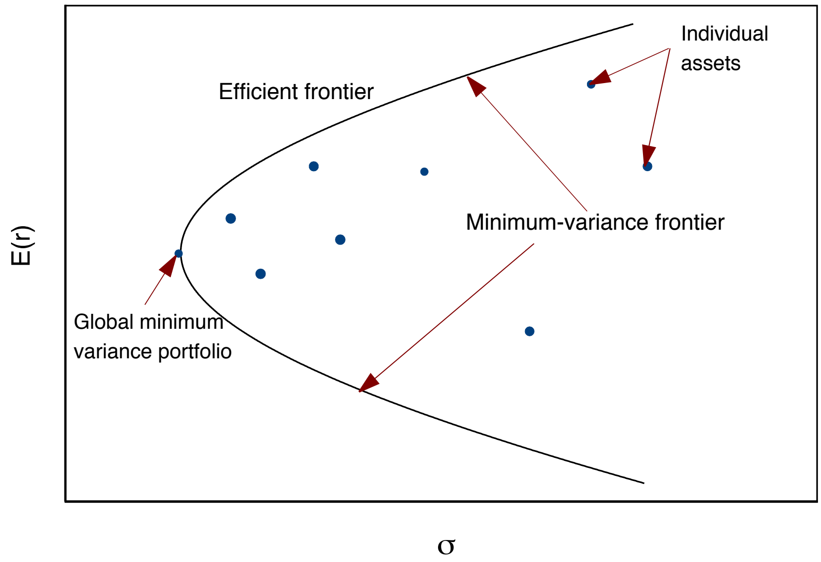
2. What are the expected return and standard deviation of the optimal risky portfolio above?

5. Markowitz Portfolio Selection Model

- We can generalize the portfolio construction problem to the case of many (n)

risky securities and a risk–free asset in the following manner

- Create an input list - you will need
  * n estimates of expected returns
  * n estimates of variances
  * $n \times (n-1)/2$ estimates of covariances
- Identify the risk-return combinations available from the set of risky assets:
  * Set the expected return of your risky portfolio to some value (for example, set $E(r_P)$ equal to 10%)
  * Consider all feasible portfolios with a given expected return that you can create out of the available risky securities. These risky portfolios are different in terms of the weights of securities composing them.
  * Choose the portfolio with the smallest standard deviation (risk) out of portfolios with the given expected return
  * Repeat the step above for other values of the expected return
  * The set of all portfolios that have minimal risk for the chosen expected returns are summarized by the **minimum-variance frontier** of risky assets. See the figure below.
  * Portfolio that has the smallest variance on the minimum variance frontier is called the **global minimum-variance portfolio**
  * all the portfolios that lie on the minimum-variance frontier, from the global minimum-variance portfolio and above, provide the best risk-return combinations; this part of the frontier is called the **efficient frontier**.
  * Question: why are portfolios on the minimum-variance frontier below the global minimum-variance portfolio inefficient?



– Note that mathematically we have to solve the following problem to find the minimal variance frontier

$$\min_{w_1, w_2, \ldots, w_n} \mathbf{w}\mathbf{V}\mathbf{w}^T,$$

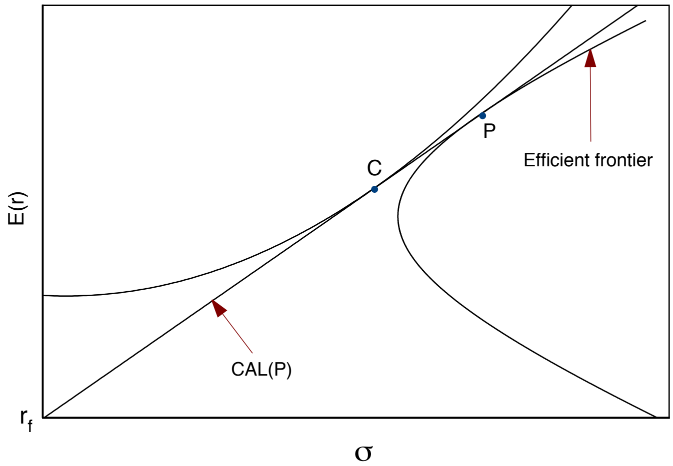
$$\text{such that} \quad E(r_P) = \sum_{i=1}^{n} w_i E(r_i) = E_0, \quad \sum_{i=1}^{n} w_i = 1$$

where $\mathbf{w}$ is an array of weights $(w_1, w_2, \ldots, w_n)$, $\mathbf{w}^T$ is a transpose array, $\mathbf{V}$ is the variance–covariance matrix, and E_0 is a chosen value of the expected return on portfolio. The term $\mathbf{w}\mathbf{V}\mathbf{w}^T$ stands for the variance of a portfolio, while the condition $\sum_{i=1}^{n} w_i E(r_i) = E_0$ is a constraint that the expected return of a portfolio should be equal to a chosen value E_0. The solution of the problem above will result in one portfolio located on the minimal variance frontier. One finds another portfolio on this frontier by changing value of E_0 and solving the problem again.

The problem above can be solved by using a software package. For example, the Excel Solver.

– After the efficient frontier has been found, we identify the optimal portfolio

P of risky assets (see the figure below) by finding the portfolio with the highest reward–to–variability ratio



  * This portfolio is given by the point of tangency between the efficient frontier and CAL passing through a riskless asset

- Finally, choose an appropriate complete portfolio C by mixing the risk–free asset with optimal risky portfolio

• Similar to the case with only two risky and one risk free securities available, the two fund separation theorem holds in the framework of Markowitz portfolio selection if all investors have the same beliefs

6. Optimal Portfolios with Restrictions on Trading

• Typically, the rate of borrowing is higher than the rate of lending. Find the solution of asset allocation problem graphically if $r_f^b > r_f^l$. Assume that the efficient frontier is known.

- First, find the steepest CAL for r_f^l and for r_f^b (see CAL(P_1) and CAL(P_2) on the figure below). Portfolio P_1 (P_2) is given by the point of tangency between the efficient frontier and CAL for r_f^l (r_f^b)

- An investor will invest in optimal risky portfolio P_1 if she lends money and in optimal risky portfolio P_2 if she borrows.

- Suppose that an investor has low risk tolerance (A is large). Then her indifference curves are steep and the optimal complete portfolio is given by point C_1 which is a point of tangency between CAL(P_1) and her indifference curve. This investor lends funds

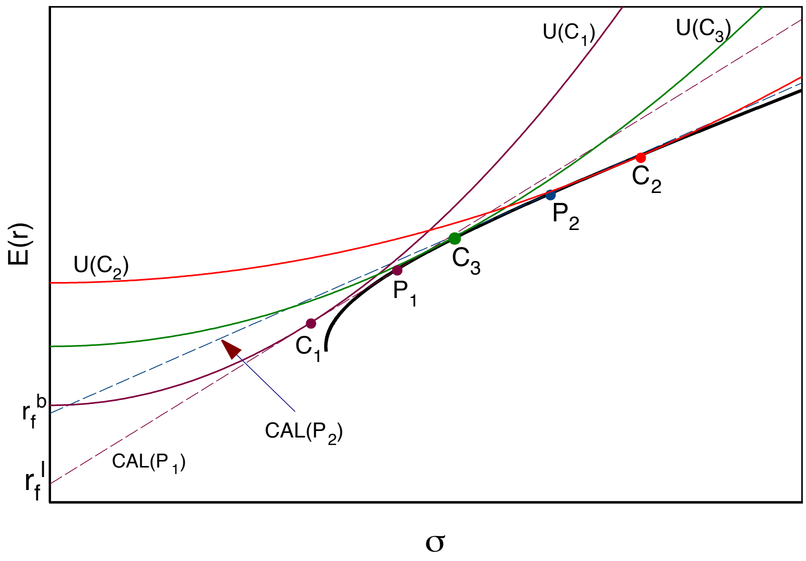
- Suppose that an investor has high risk tolerance (A is small). Then her indifference curves have small slope and the optimal complete portfolio is given by point C_2 which is a point of tangency between CAL(P_2) and her indifference curve. This investor borrows funds

- Suppose that an investor has moderate risk tolerance (A is neither small nor large). Then her indifference curves have medium slope and the optimal complete portfolio is given by point C_3 which is a point of tangency between the efficient frontier and her indifference curve. This investor neither borrows nor lends funds

- Now let us outline the algorithm for finding optimal complete portfolio C in the case when our risky portfolio can include only two risky securities K and L. We consider an investor with a given coefficient of risk aversion A.

  - Assume that an investor lends. She buys portfolio P_1 and T–bills. Find the weights of securities K and L in portfolio P_1 by using the formula for w_2^* on page 40. Then find the expected return and variance of P_1. Next find the weight y^* of portfolio P_1 in complete portfolio C_1. If $y^* \leq 1$ then problem is solved.

  - If $y^* > 1$ then we arrive at contradiction with our original assumption. Therefore, assume that an investor borrows. She buys portfolio P_2 and



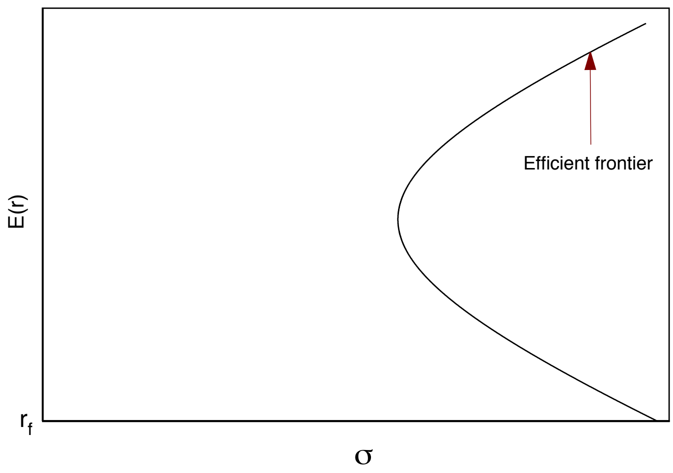
sell T–bills. Find the weights of securities K and L in portfolio P_2 by using the formula for w_2^* on page 40. Next find the weight y^* of portfolio P_2 in complete portfolio C_2. If $y^* > 1$ then problem is solved.

– If $y^* \leq 1$ then we arrive at contradiction with our last assumption. Therefore, an investor neither borrows nor lends and buys portfolio C_3. Find the weights of securities K and L in portfolio $P_3 = C_3$ by using the formula for w_2^* on page 36.

Practice Problem

Find the solution of asset allocation problem graphically for two investors if borrowing is not allowed. Assume that the efficient frontier is known and one investor has very low risk tolerance while the other has very high risk tolerance (see the figure below).

Question: How do you find the optimal portfolio for an investor with very high risk tolerance algebraically when the market has only two risky securities K and L?

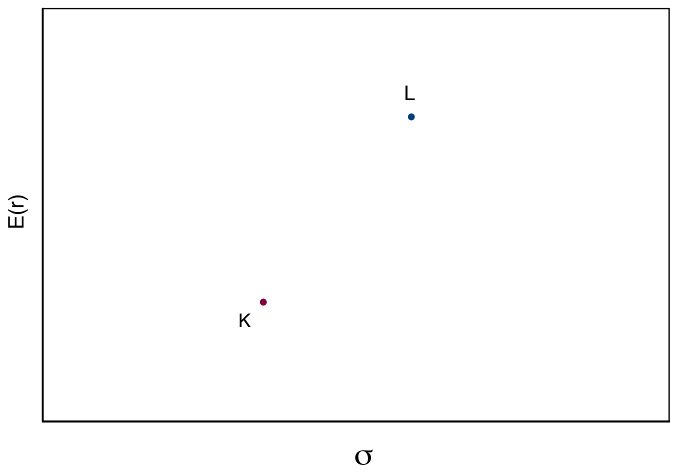


Practice Problem

Suppose that you have two stocks K and L. Stocks cannot be sold short.

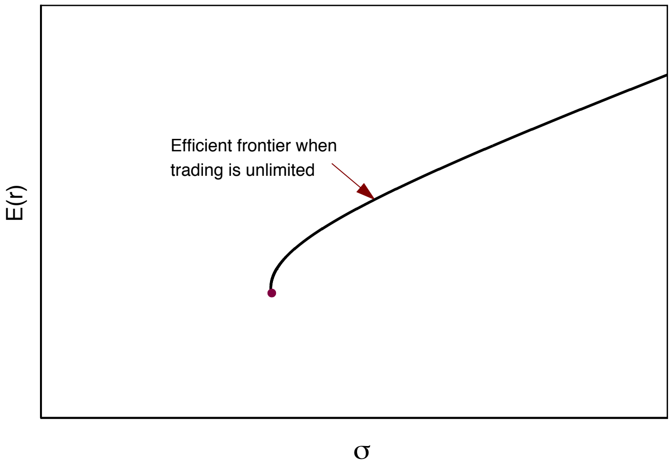
What would the efficient frontier look like in this case?

What would be the efficient frontier if you could sell short stock K, but not stock L?



Practice Problem

Assume that there are a lot of risky assets that can be traded with no restrictions. Moreover, their efficient frontier is known and each portfolio on this frontier includes a short position in at least one risky security. Sketch the efficient frontier for these risky assets in assumption that their short sales are not allowed



Additional Practice Problems

1. An investor invests 30% of his wealth in a risky asset with an expected rate of return of 0.15 and a variance of 0.04 and 70% in a T-bill that pays 6%. His portfolio's expected return and standard deviation are

A) 0.114 and 0.12

B) 0.087 and 0.06

C) 0.295 and 0.12

D) 0.087 and 0.12

E) none of the above,

respectively

2. You invest \$1000 in a complete portfolio which is comprised of a risky asset with an expected rate of return of 12% and a standard deviation of 20% and a T-bill with a rate of return of 5%. If you want your complete portfolio to have a standard deviation of 10%, how much should you invest in T-bills? What would the expected return on that portfolio be?

3. You have $\$1,000,000$ to invest. The risk-free rate as well as your borrowing rate is 2%. The return on the risky portfolio is 12%. If you wish to earn a 22% return, how much should you borrow?

5. You are considering investing $\$1,000$ in a complete portfolio. The complete portfolio is comprised of T-bills that pay 3% and a risky portfolio P, constructed from 2 risky securities X and Y. The weights of X and Y in P are 80% and 20%, respectively. X has expected rate of return of 15% and Y has an expected rate of return of 8%. To form a complete portfolio with an expected rate of return of 10%, how much should you invest in T -bills, security X and security Y?

6. An investor can design a risky portfolio based on two stocks, K and L. Stock

K has an expected rate of return of 18% and a standard deviation of return of 30%. Stock L has an expected rate of return of 14% and a standard deviation of return of 5%. The correlation coefficient between the two stocks is 0.5. The risk-free rate is 5%.

(a) what is the weight of stock K in the optimal risky portfolio?

(b) what is the expected return on the optimal risky portfolio?

(c) what is the standard deviation of return on the optimal risky portfolio?

7. When borrowing and lending at a risk-free rate are allowed, which Capital Allocation Line (CAL) should the investor choose to combine with the efficient frontier?

I) with the highest reward-to-variability ratio.

II) that will maximize his utility

III) with the steepest slope.

IV) with the lowest slope.

A) I and III

B) I and IV

C) II and IV

D) I only

E) I, II, and III